On the Computation of Equilibria in Discrete First-Price Auctions

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Abstract

The **first-price auction** is arguably the simplest auction format, where there is one good for sale, which gets allocated to the highest bidder for a payment equal to her bid. Contrary to the second-price auction, bidders are incentivized to **strategize** depending on their beliefs about the others, leading to a game of **incomplete information**. We are interested in stable states (equilibria) of such games, in which bidders are not incentivized to deviate from their strategy. Filos-Ratsikas et al. (2023) recently studied the **computational complexity** of finding these equilibria, proving PPAD-completeness of the problem in the setting of subjective, continuous priors. Follow-up work in [Chen and Peng 2023] established a PPAD-completeness result in the common priors setting, under a trilateral tie-breaking rule. The setting of discrete distributions was left as an open problem in the above works. The choice of **discrete values** is both logical and practical, considering the common assumption that values are monetary, as well as the representation of strategies. In this work, we settle the complexity of equilibrium computation in the setting of **discrete, subjective priors**. We also provide new results for other settings that can accelerate future work.

First-Price Auction Setting

- Set of bidders $N = \{1, 2, \dots n\}$
- Discrete value space and bidding space $V, B \subset [0, 1]$
- Pure strategy: $\hat{\beta}_i : V \to B$
- Ex-post utility:

$$\tilde{u}_i(\boldsymbol{b}; v_i) := \begin{cases} \frac{1}{|W(\mathbf{b})|}(v_i - b_i), & \text{if } i \in W(\mathbf{b}), \\ 0, & \text{otherwise,} \end{cases} \quad \text{where } W(\boldsymbol{b}) = \operatorname*{argmax}_{j \in N} b_j$$

Prior Beliefs

Each bidder i has (subjective) prior belief $F_{i,j}$ for bidder j.

- Independent Private Values (Common Priors): $F_{i,j} = F_{i',j}, \quad \forall i, i' \in N \setminus \{j\}$
- Identical Independent Values (iid): bidder values are iid according to some distribution F.

- Expected utility: $u_i(b, \hat{\boldsymbol{\beta}}_{-i}; v_i) := \mathbb{E}_{\boldsymbol{v}_{-i} \sim \boldsymbol{F}_{-i}}[\tilde{u}_i(b, \hat{\boldsymbol{\beta}}_{-i}(\boldsymbol{v}_{-i}); v_i)]$

This is a game of incomplete information $\Rightarrow$ consider *Bayes-Nash Equilibria*.

Pure Bayes-Nash Equilibrium

A strategy profile $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1, \dots, \hat{\beta}_n)$ is an ε-approximate Pure Bayes-Nash Equilibrium (PBNE) if for any bidder $i \in N$, any value $v_i \in V$, and any bid $b \in B$:

$$u_i(\hat{\beta}_i(v_i), \hat{\boldsymbol{\beta}}_{-i}; v_i) \geq u_i(b, \hat{\boldsymbol{\beta}}_{-i}; v_i) - \varepsilon$$

i.e., no bidder can increase her utility by more than ε by unilaterally deviating. Strategies are assumed to be *no-overbidding*.

Result 1: NP-completeness for PBNE

The problem of deciding the existence of a Pure Bayes-Nash Equilibrium in a first-price auction with discrete subjective priors and discrete bids is **NP-complete**.

Mixed Strategies

The hardness of computing a PBNE motivates the study of mixed strategies.
Mixed strategy: $\beta_i : V \to \Delta(B)$ (distribution over bids)
Solution Concept: (ε-approximate) Mixed Bayes Nash Equilibrium

Mixed Bayes-Nash Equilibrium

A strategy profile $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ is an ε-approximate Mixed Bayes-Nash Equilibrium (MBNE) if for any bidder $i \in N$, any value $v_i \in V$, and any distribution over bids $\boldsymbol{\gamma} \in \Delta(B)$:

$$u_i(\beta_i(v_i), \boldsymbol{\beta}_{-i}; v_i) \geq u_i(\boldsymbol{\gamma}, \boldsymbol{\beta}_{-i}; v_i) - \varepsilon$$

Equivalence Result

We present an equivalence result between PBNE in the *continuous setting* and MBNE in the discrete setting. This can be used to translate results between the two settings.

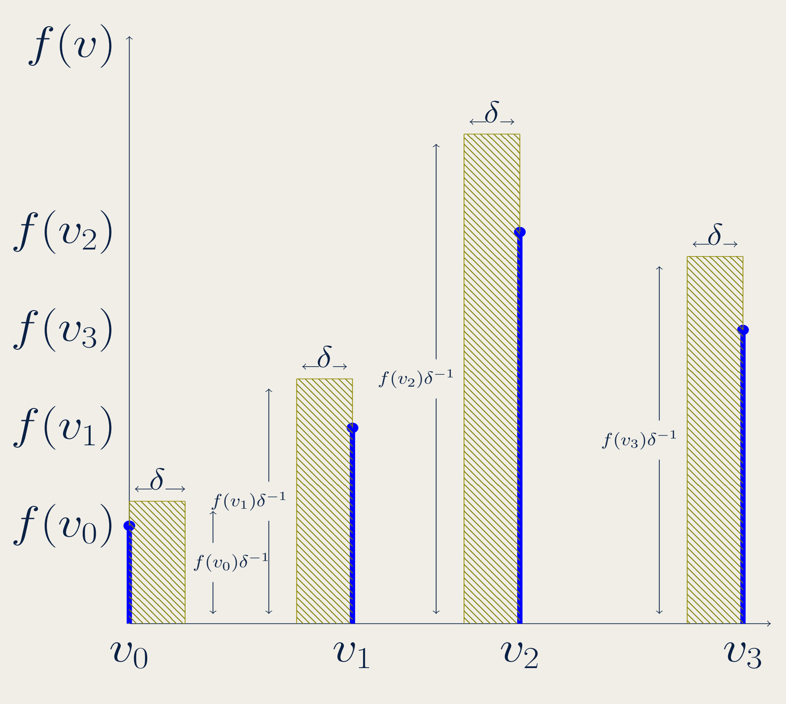

Figure 1. Discrete → Continuous

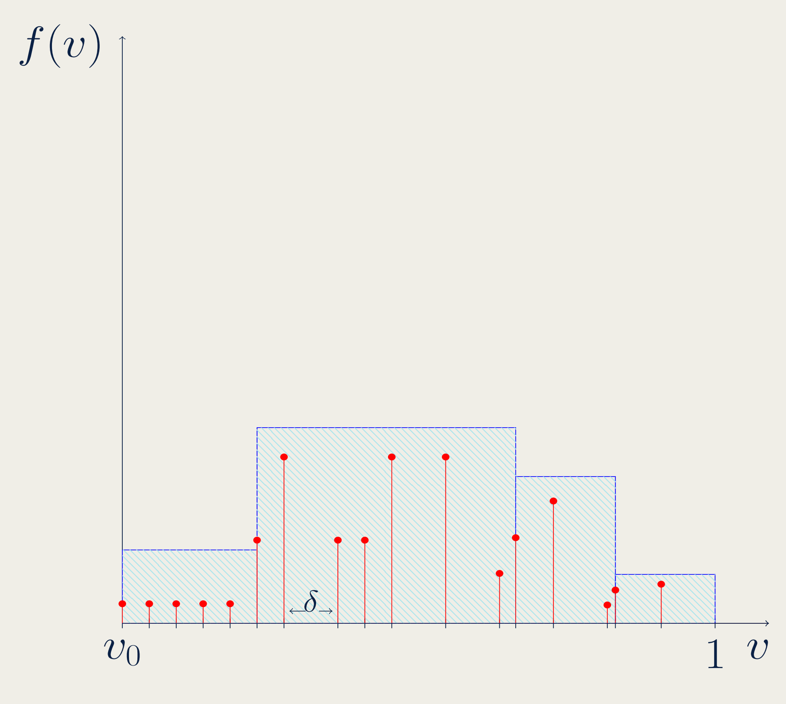

Figure 2. Continuous → Discrete

The Complexity of Computing a MBNE

Previous work in [Filos-Ratsikas et al. 2023] shows that the problem is PPAD-complete in the continuous setting.

Using our equivalence result, we can translate this to a proof of existence of a MBNE in the discrete setting, yielding PPAD-membership of the problem.

We provide a reduction from the PPAD-complete problem PURECIRCUIT [Deligkas et al. 2022] to establish the following result:

Result 2: PPAD-completeness of computation of MBNE

The problem of computing a Mixed Bayes-Nash Equilibrium of a first-price auction with discrete subjective priors and discrete bids is **PPAD-complete**.

The IID Setting

In the iid setting it makes sense to consider **symmetric equilibria**, where all bidders play the same strategy.

Idea: MBNE constraints can be written as a system of polynomial inequalities (with the number of terms being exponential in $N, |B|, |V|$). To guarantee the feasibility of the system, we provide the first proof of existence of a symmetric MBNE for iid priors.

1. We prove the existence of a symmetric PBNE in the continuous setting, using Kakutani's fixed point theorem [Kakutani 1941].
2. We translate the result to the discrete setting using our equivalence result.

To succinctly represent the aforementioned system of polynomial inequalities, we carry out the following steps:

1. Use **symmetry** to remove the exponential dependency on N.
2. **"Shrinkage Lemma"**: Restrict the bidding space to B' of size $O(1/\varepsilon)$, show how to translate an ε-PBNE in the shrunk space to an ε'-PBNE in the original bidding space.
3. Use **monotonicity** to succinctly represent strategies.

Finally, we find a δ-near solution to the **system of polynomial inequalities** using a known result from [Grigor'ev and Vorobjov 1988] and **round** the solution back to a feasible set of strategies.

Result 3: PTAS for iid

The problem of computing a symmetric ε-approximate Mixed Bayes-Nash Equilibrium of a first-price auction with iid priors admits a **PTAS**.

Future Work

- What is the complexity of computing a PBNE in either the continuous or the discrete Independent Private Values setting with uniform tie-breaking?
- Can we design similar efficient algorithms for more general settings?

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