Equilibrium Computation in First-Price Auctions with Correlated Priors

Aris Filos-Ratsikas[1], Yiannis Giannakopoulos[2], Alexandros Hollender[3], **Charalampos Kokkalis[1]**

[1]University of Edinburgh, [2]University of Glasgow, [3]University of Oxford

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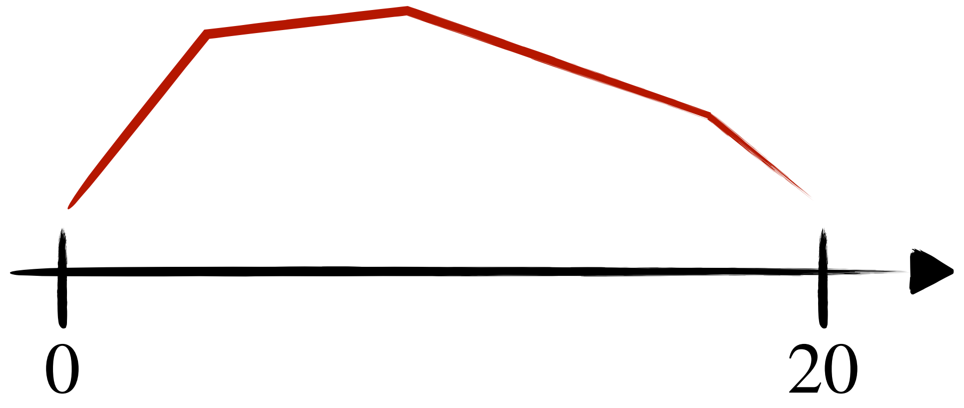
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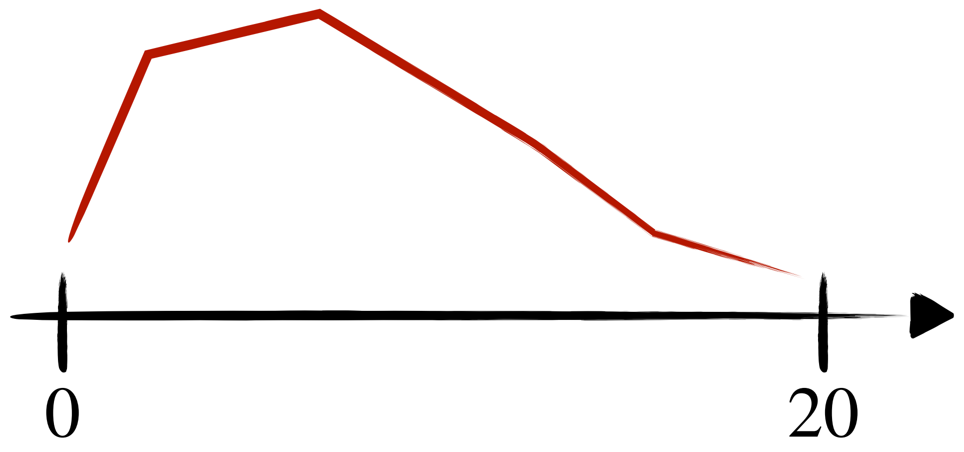
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What are these distributions?

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These can be continuous (CFPA) or discrete (DFPA).

Bayes-Nash Equilibrium

- A strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ is an ε-approximate pure Bayes-Nash Equilibrium if for any bidder $i \in N$, any value $v_i \in V$, and any bid $b \in B$:

$$u_i(\beta_i(v_i), \beta_{-\mathbf{i}}; v_i) \geq u_i(b, \beta_{-\mathbf{i}}; v_i) - \varepsilon$$

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- If β also satisfies $\beta_i = \beta_j$, for all $i, j \in N$, then the equilibrium is symmetric.

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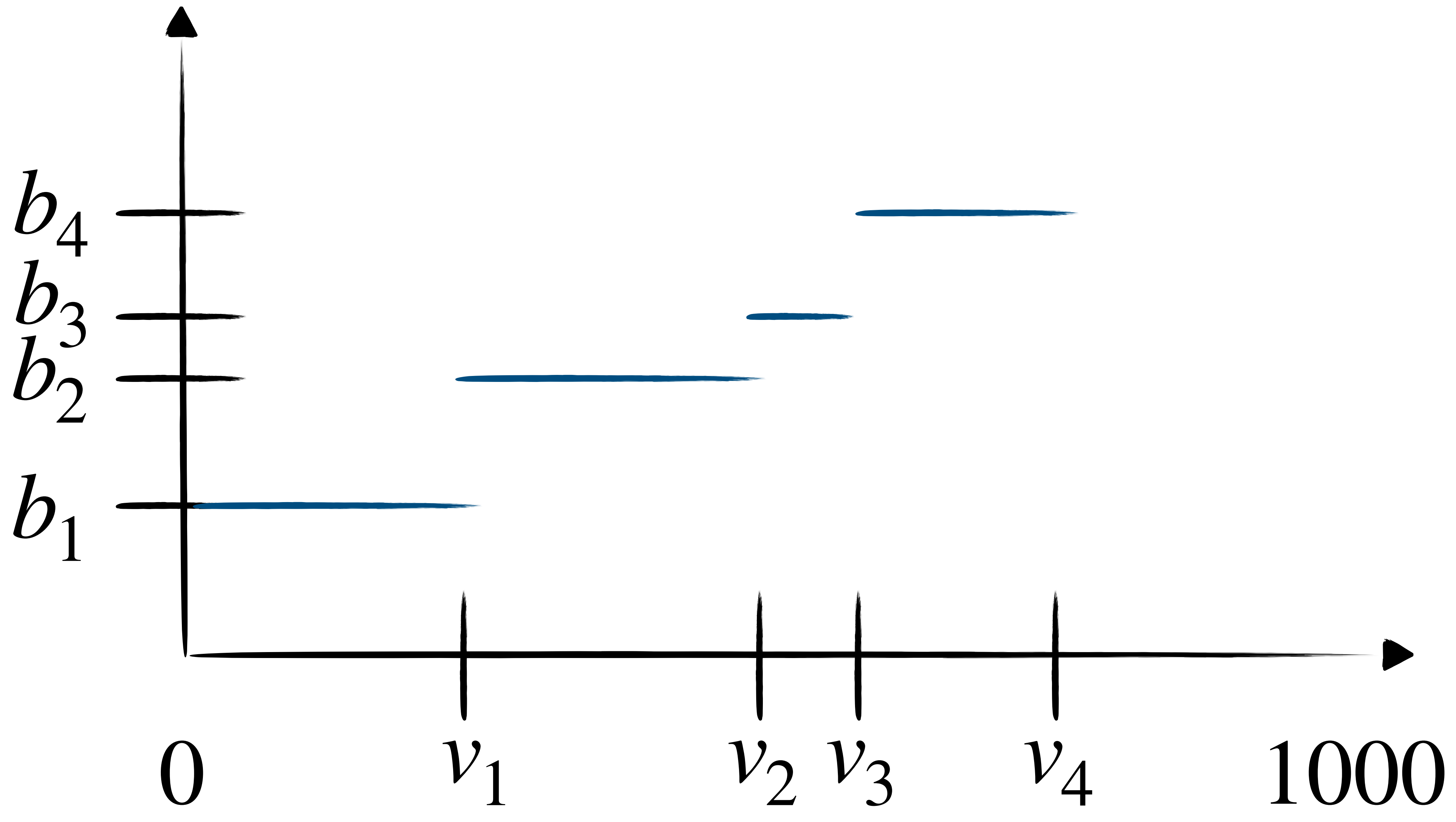
- Jump-point representation [Athey'01]: provide the points in the value space where the strategy "jumps" to the next bid.



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- In the DFPA with *subjective priors*, the decision problem was shown to be NP-hard [Filos-Ratsikas et al.'24].

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What about mixed equilibria?

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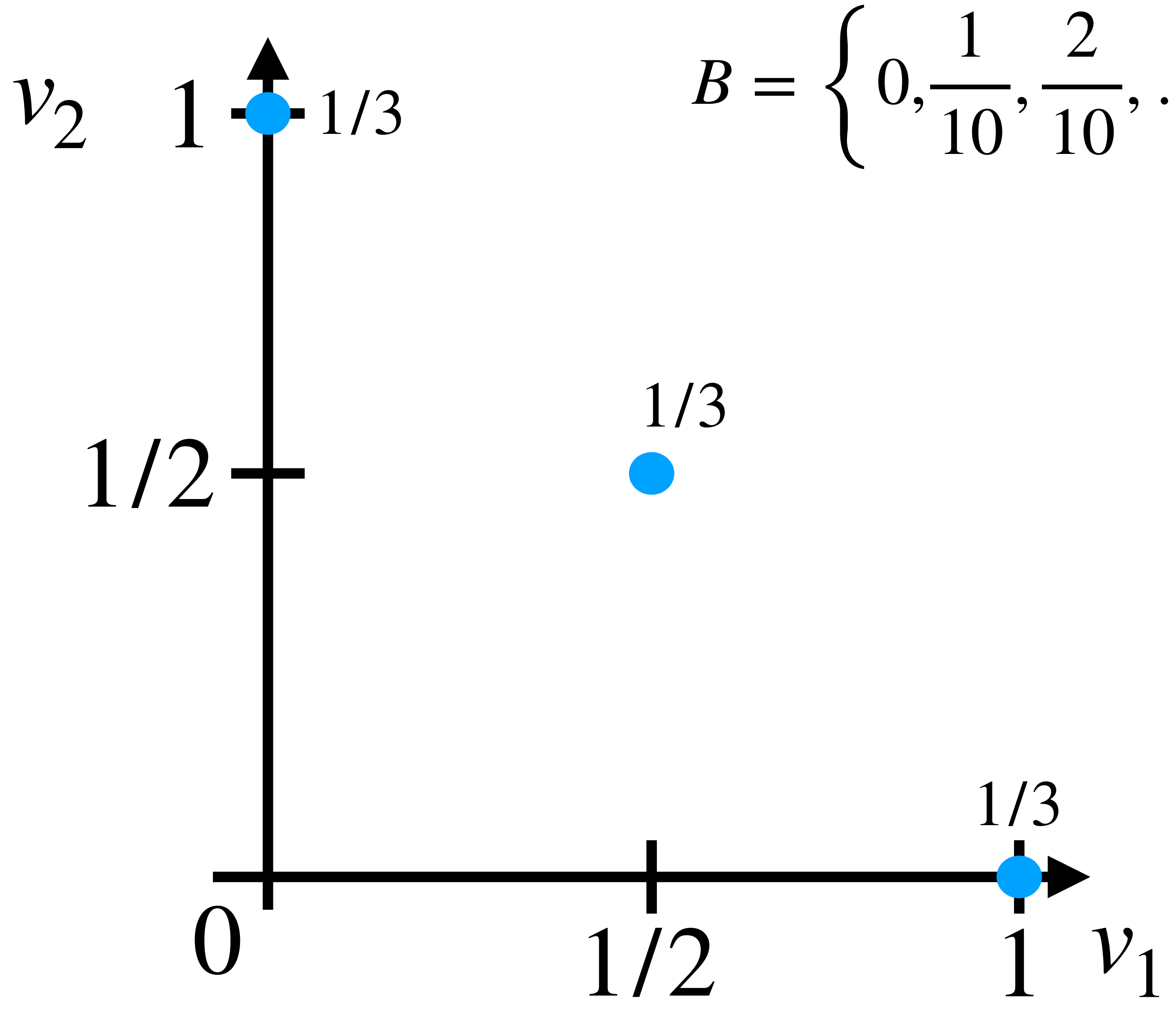
- Mixed strategies restore continuity $\Rightarrow$ existence of an MBNE (Bayesian games)
- **Question:** Does a monotone MBNE exist?

Monotone MBNE in Correlated Priors

Theorem [this work]**:** There are instances of the DFPA with correlated values which admit no monotone MBNE.

Counterexample:



$$B = \left\{0, \frac{1}{10}, \frac{2}{10}, \ldots, 1\right\}$$

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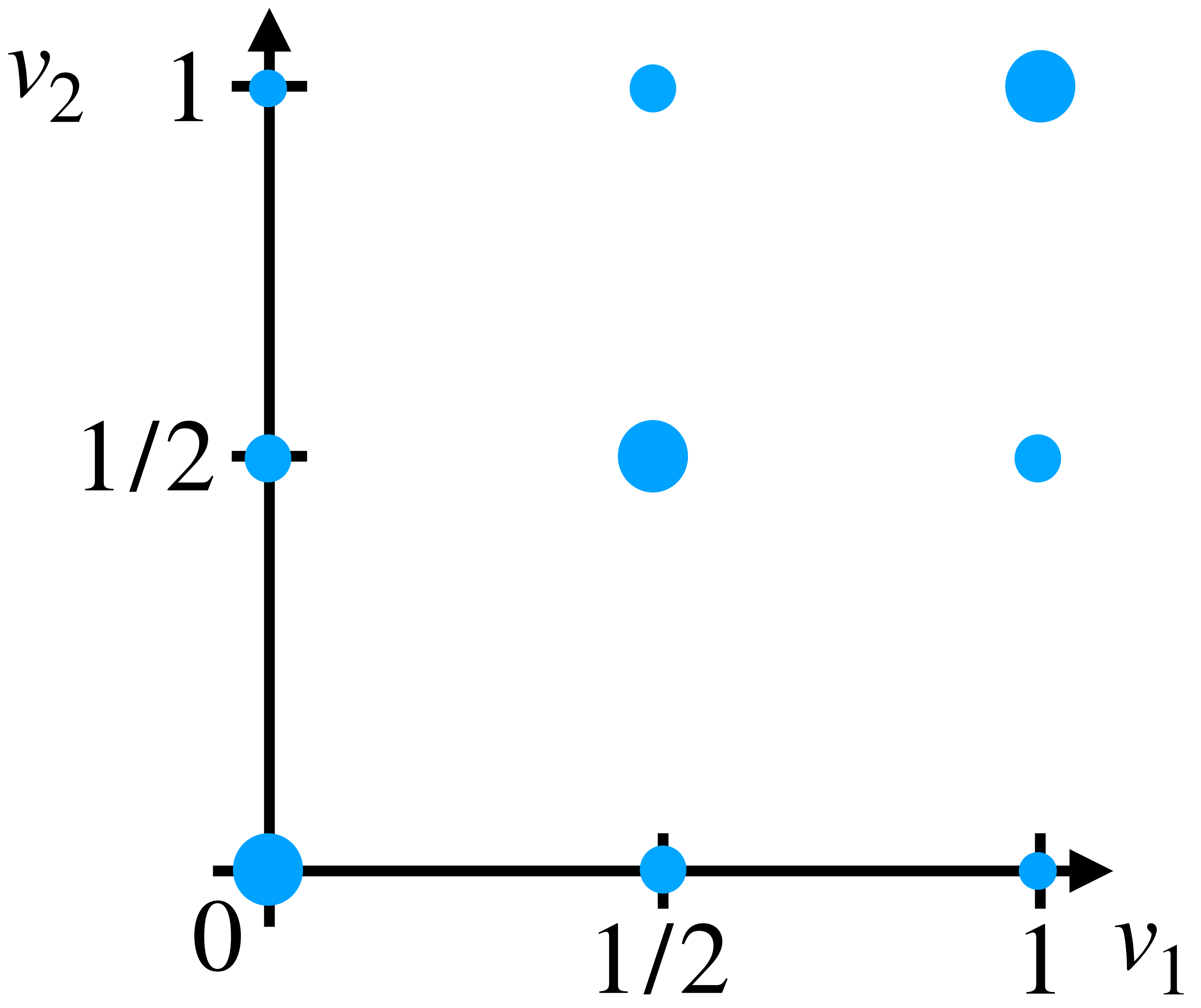

Affiliation condition:

$$f(v \vee v') \cdot f(v \wedge v') \geq f(v) \cdot f(v')$$

Positive Results

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$$\text{where} \quad L_v(y) := \exp\left(- \int_y^v \frac{g_t(t)}{G_t(t)}\,\mathrm{d}t \right) \quad \forall y \in [0,v]$$

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But what about computation?

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Issue #1: Continuous bidding space

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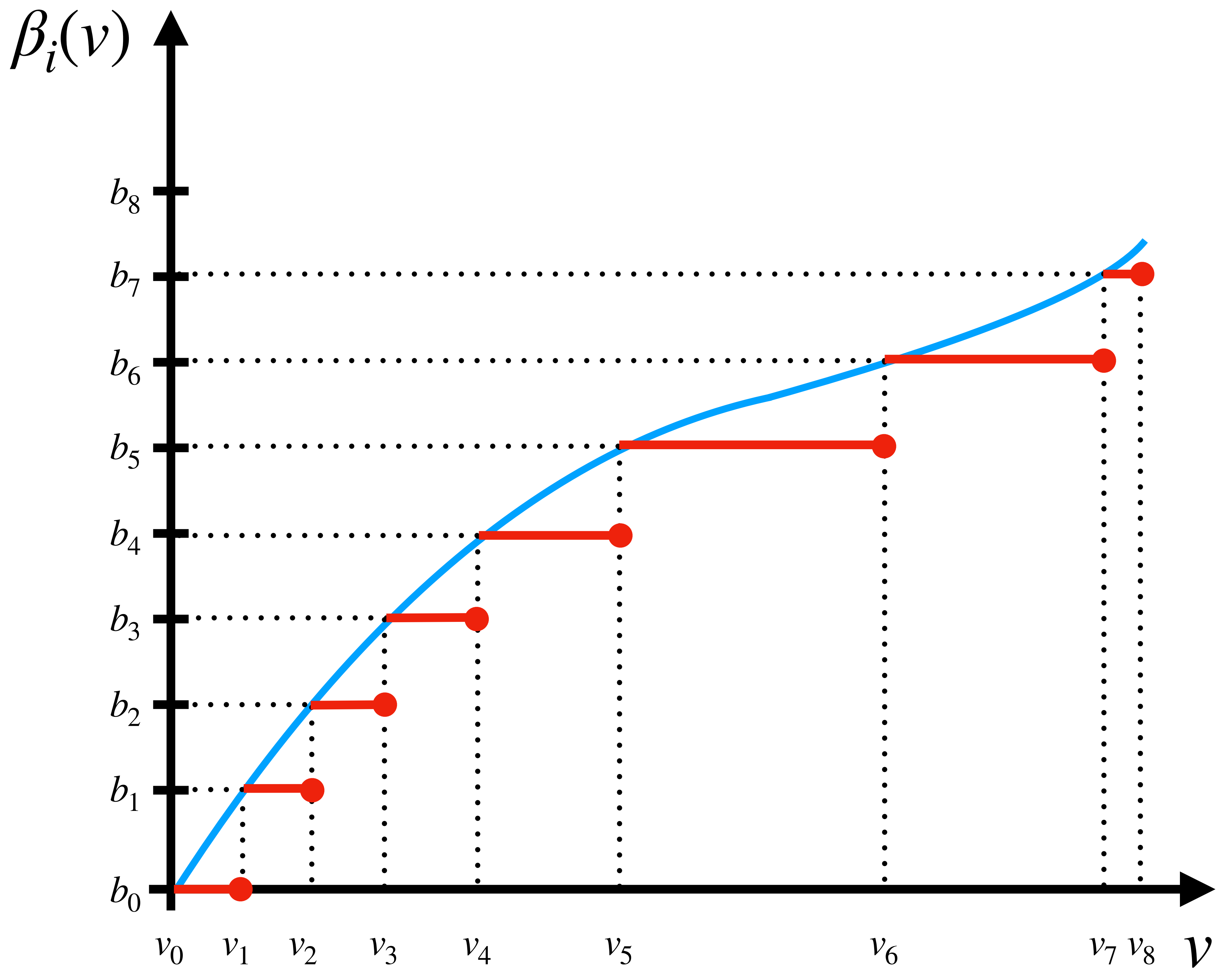
Issue #2: Computation of integrals

Solution: Approximate again, using properties of the distributions.

Bid Densification



Theorem: Consider a CFPA with n bidders and symmetric APV and δ-dense bidding space.

Then, we can compute an ε-approximate* PBNE in time polynomial in the problem description and $\log(1/\varepsilon)$, if either of the following holds:

i) f is (ϕ_1, ϕ_2)-bounded, i.e., $\phi_1 \leq f(x) \leq \phi_2$, $\forall x \in [0,1]$

ii) The values are i.i.d.

*approximation depends polynomially on $\delta, n, \dfrac{\phi_2}{\phi_1}$

Our Contribution and Future Work

Computational Complexity

	i.i.d.	SAPV	IPV	APV	Correlated
CFPA	FPTAS* for δ-dense B	(Symmetric) PBNE: PTAS FPTAS* for δ-dense B		PTAS for constant n	
DFPA	(Symmetric) MBNE: PTAS	(Symmetric) MBNE: PTAS		MBNE: PTAS for constant n	PBNE: NP-complete

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DFPA	(Symmetric) MBNE: PTAS	(Symmetric) MBNE: PTAS	**MBNE?** **PBNE?**	MBNE: PTAS for constant n	PBNE: NP-complete

Thank you!

Questions? charalampos.kokkalis@ed.ac.uk

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