On the Computation of Equilibria in Discrete First-Price Auctions

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25TH ACM CONFERENCE ON ECONOMICS & COMPUTATION

EC24

NEW HAVEN

THE UNIVERSITY of EDINBURGH

University of Glasgow

UNIVERSITY OF OXFORD


HIYA PALS!


lemon bucket orkestra


fringe
Make it happen
RBS
The Royal Bank of Scotland
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Strategic environment induces game between bidders!

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First-Price Auctions

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- Pure strategy: $\beta_i : V \to B$
- Ex-post utility: $\tilde{u}_i(\mathbf{b}; v_i) := \begin{cases} \frac{1}{|W(\mathbf{b})|}(v_i - b_i), & \text{if } i \in W(\mathbf{b}), \\ 0, & \text{otherwise,} \end{cases}$ where $W(\mathbf{b}) = \text{argmax}_{j \in N} b_j$

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Solution concept? Bayes-Nash Equilibrium

Bayes-Nash Equilibrium

- A strategy profile $\beta = (\beta_1, \ldots, \beta_n)$ is an ε-approximate pure Bayes-Nash Equilibrium if for any bidder $i \in N$, any value $v_i \in V$, and any bid $b \in B$:

$$u_i(\beta_i(v_i), \beta_{-\mathbf{i}}; v_i) \geq u_i(b, \beta_{-\mathbf{i}}; v_i) - \varepsilon$$

- At an equilibrium, no bidder wants to unilaterally change strategy.

We refer to a 0-approximate PBNE as an *exact* PBNE.

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2. Can we compute it efficiently?

- [FGHLP23] introduced computational study of the problem, showed PPAD-completeness in the continuous, subjective prior setting.

- Follow up work in [CP23] proved PPAD-completeness of the problem in the continuous common priors setting (under a “trilateral” tie-breaking rule).

Prior Work

continuous priors

PBNE (trilateral tie-breaking): PPAD-complete [CP23]

PBNE: PPAD- and FIXP-complete [FGHLP23]

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iid priors — common priors — subjective priors

- Prior work left the setting of discrete distributions as an open problem.

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- Discrete bidding space
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- Discrete distributions ⇒ Existence of equilibria is not guaranteed (e.g., see [EMS09])
- **Question:** Could the problem be easier in the discrete setting?

Main Results

1. NP-completeness of deciding the existence of a Pure Bayes-Nash Equilibrium in a DFPA with subjective priors

2. PPAD-completeness of computing a Mixed Bayes-Nash Equilibrium in a DFPA with subjective priors

3. PTAS for computing a symmetric Mixed Bayes-Nash Equilibrium when the priors are iid

Existence of ε-PBNE

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1. NP membership: Compute a bidder's expected utility given a strategy profile and her value using dynamic programming, use it to verify certificates.

2. NP-hardness: Reduce from the CIRCUIT-SAT problem.

Mixed Strategies

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Reminder

PPAD: class containing problems where existence is guaranteed due to a parity argument on directed graphs (e.g., NASH)

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- Computing an ε-MBNE in a DFPA is a Total Search Problem.
- **Question:** What is the appropriate complexity class for this problem? PPAD?
- **Idea:** Connection between mixed equilibria in the discrete setting and pure equilibria in the continuous setting.

Equivalence Result

Discrete

Continuous

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DFPA,
$\delta \in (0,1)$

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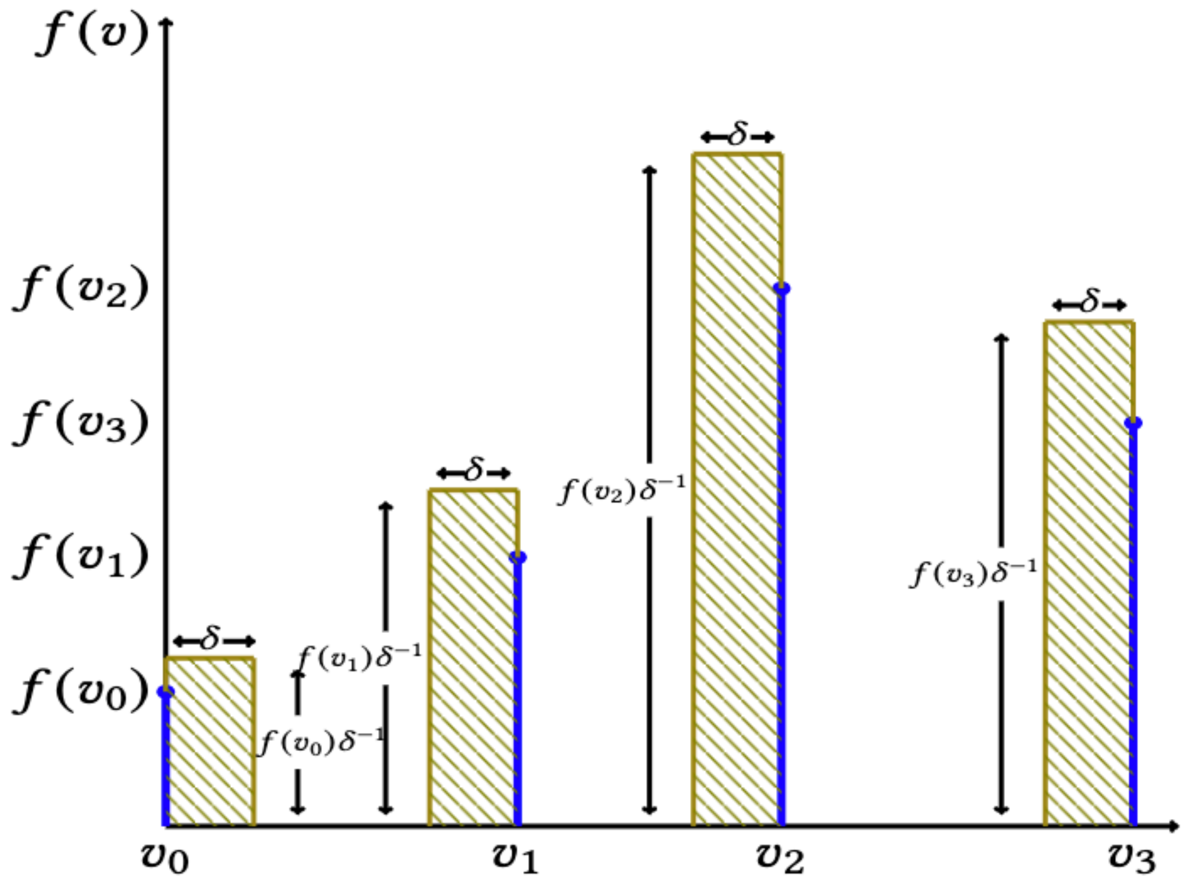

Figure 1: Discrete $\longrightarrow$ Continuous

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DFPA,
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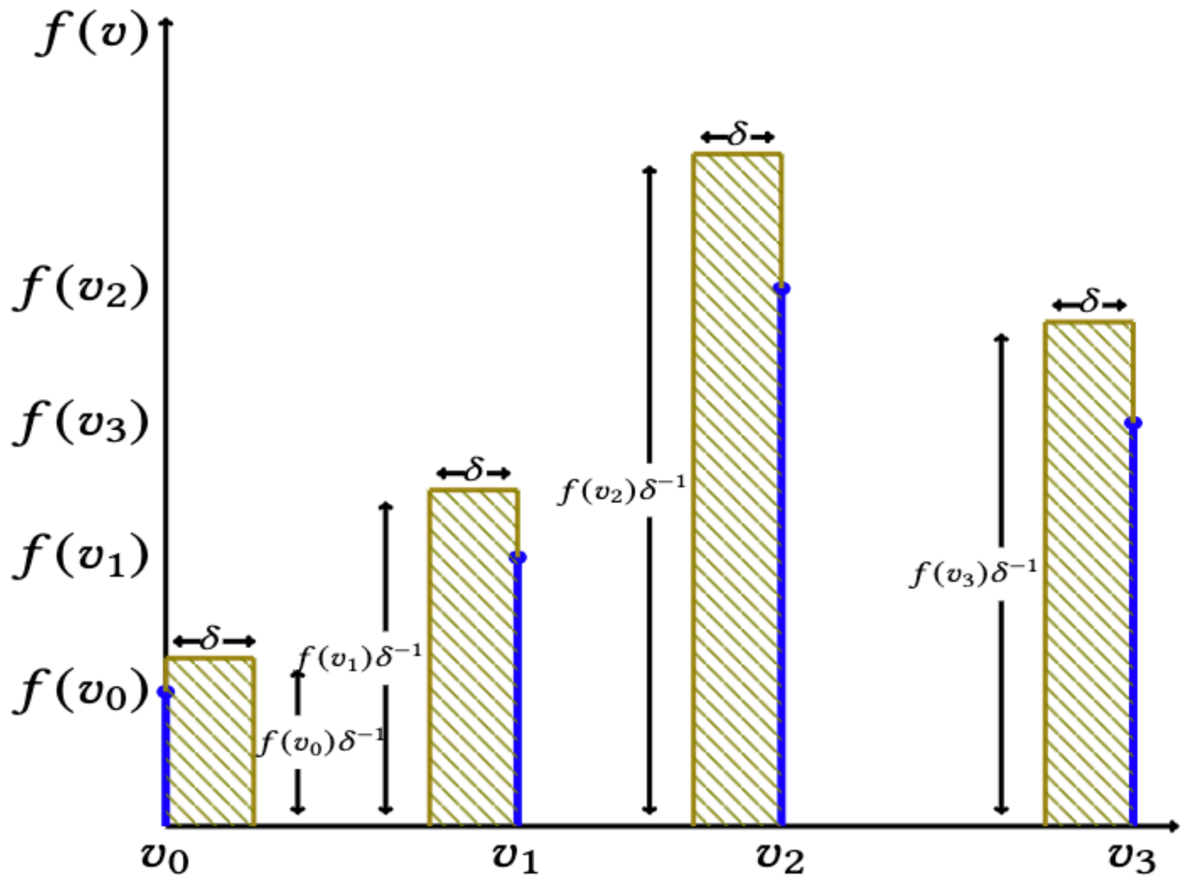

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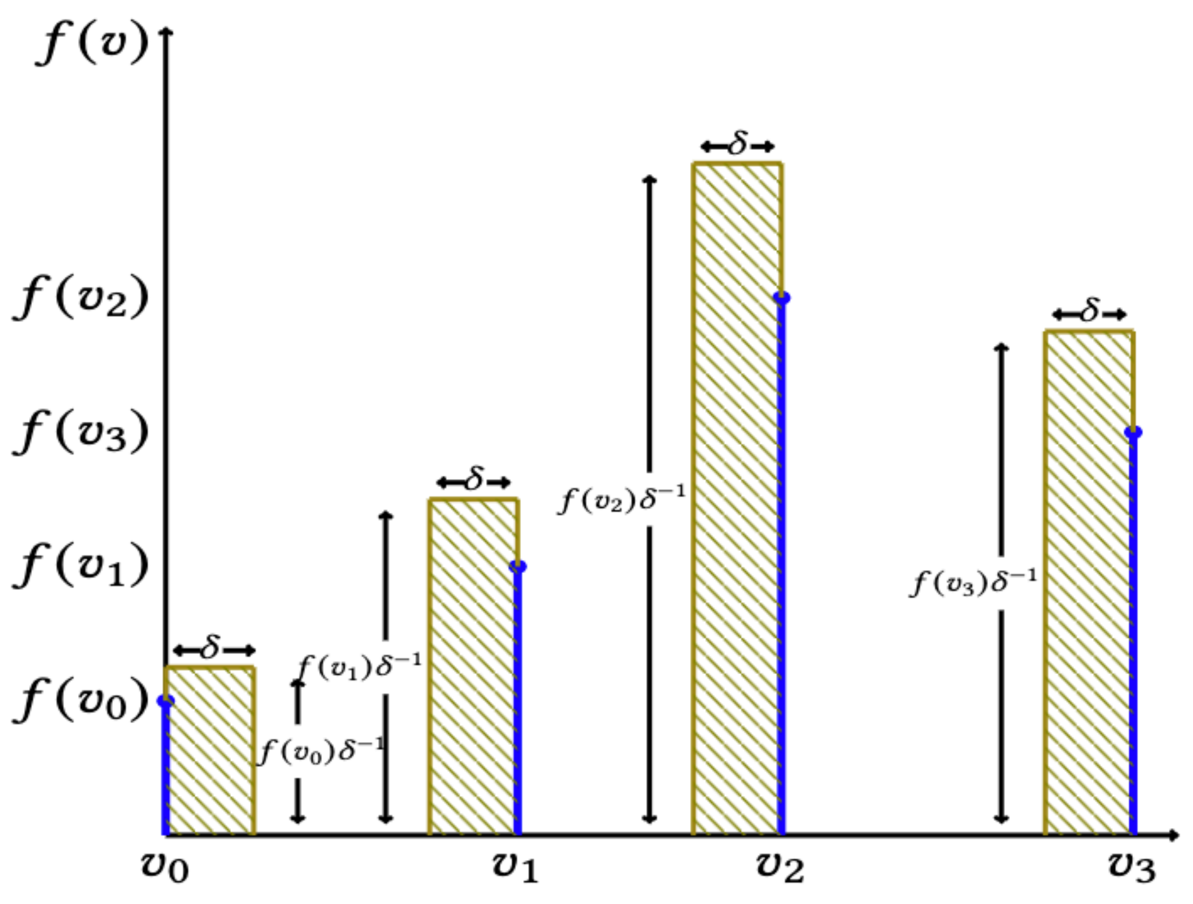

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CFPA

ε-PBNE,
$\forall \varepsilon \geq 0$

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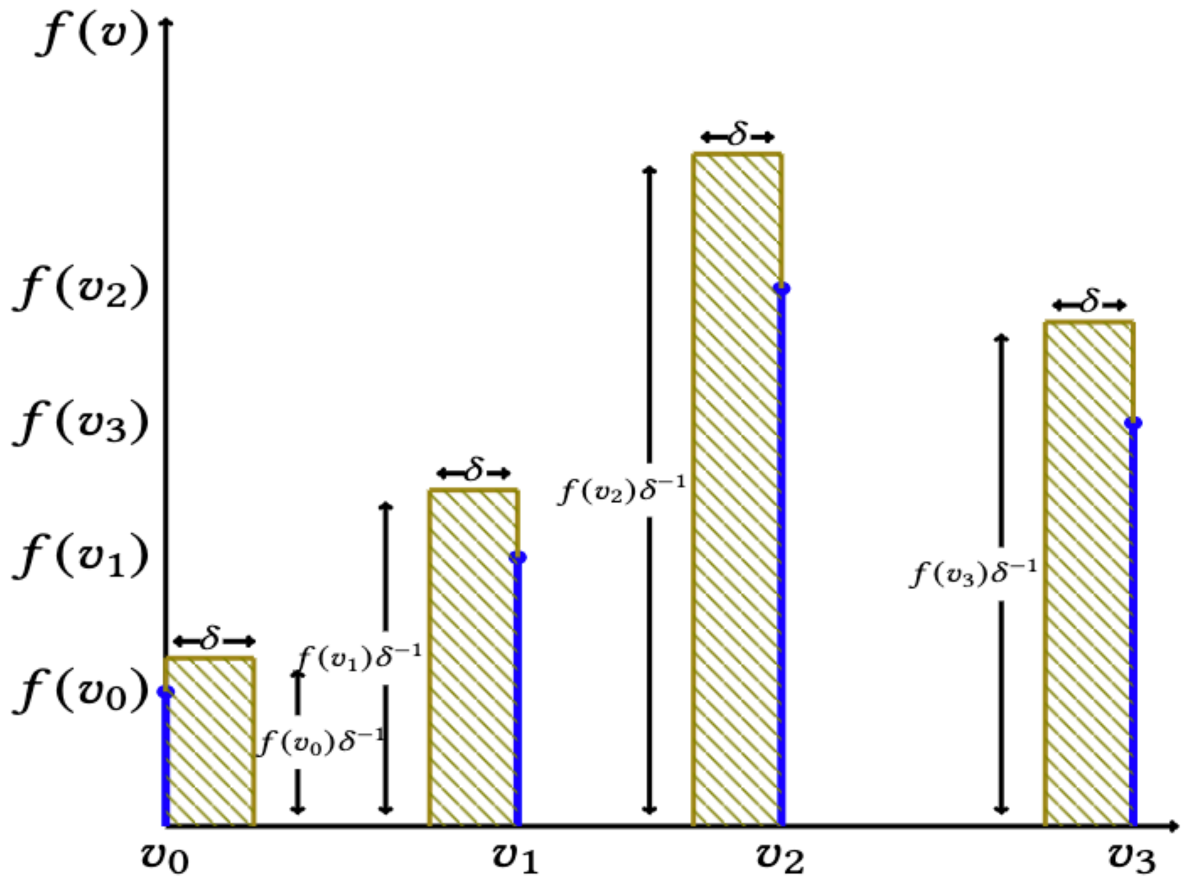

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$(\varepsilon + \delta)$-MBNE

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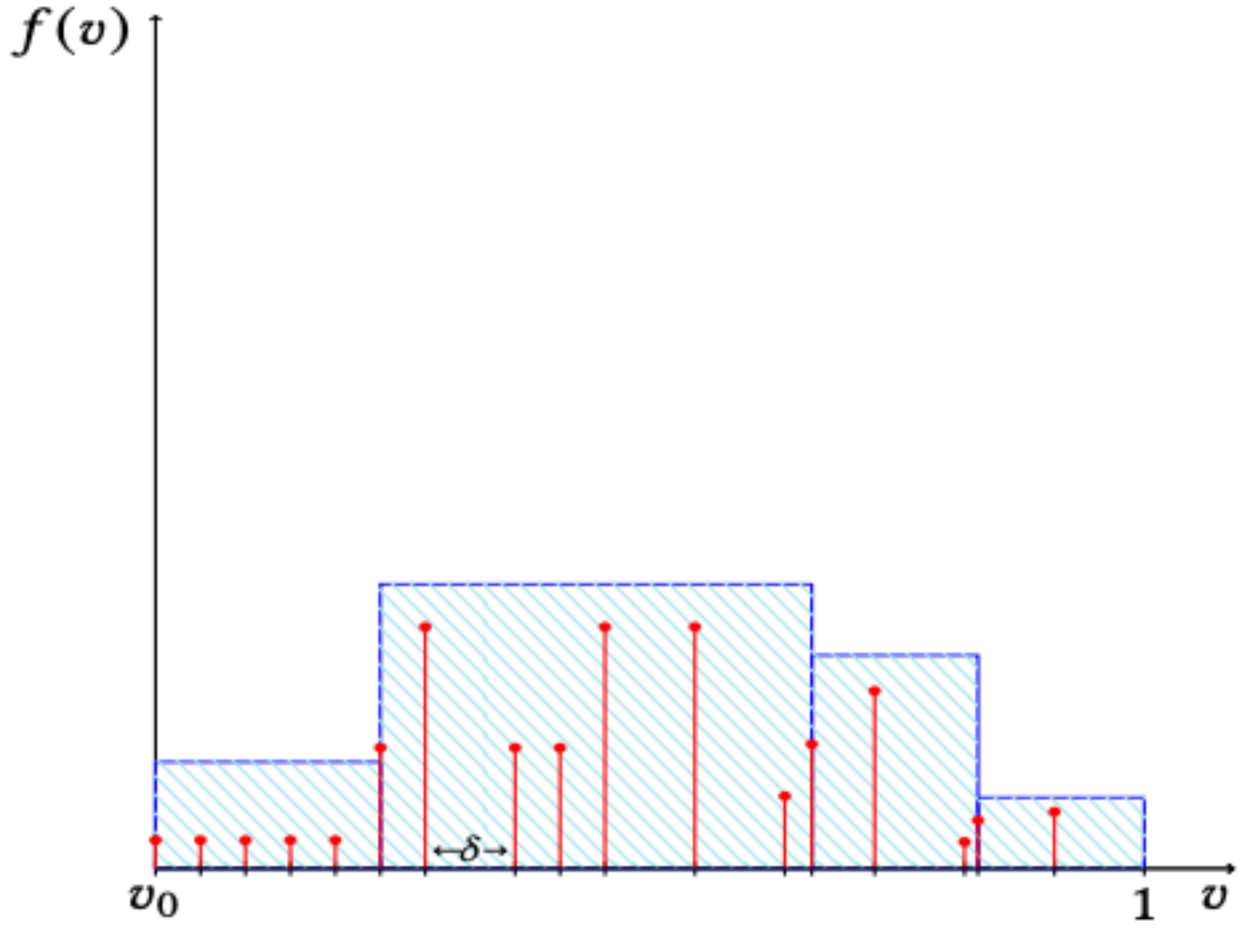

Figure 2: Continuous ⟶ Discrete

Equivalence Result

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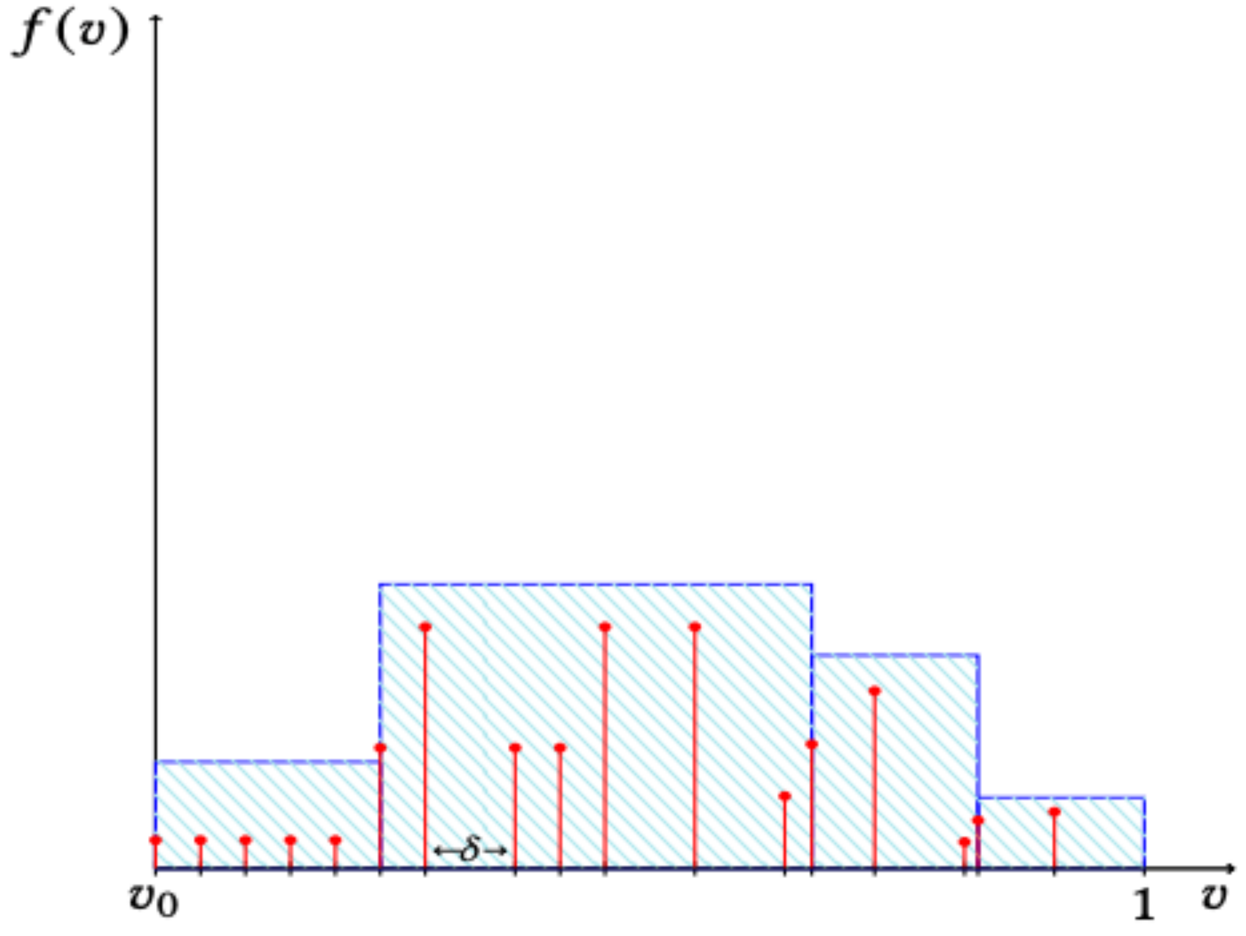

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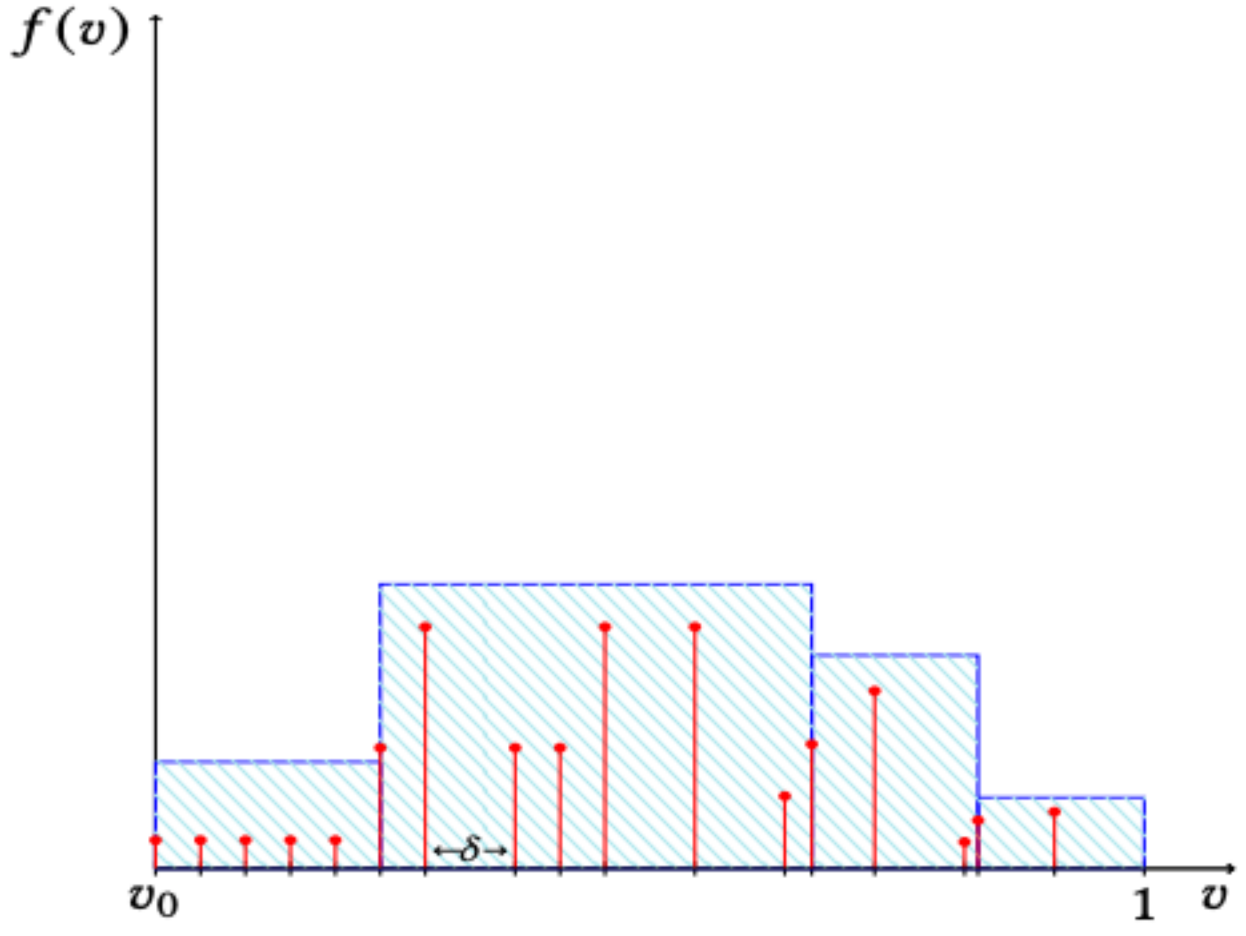

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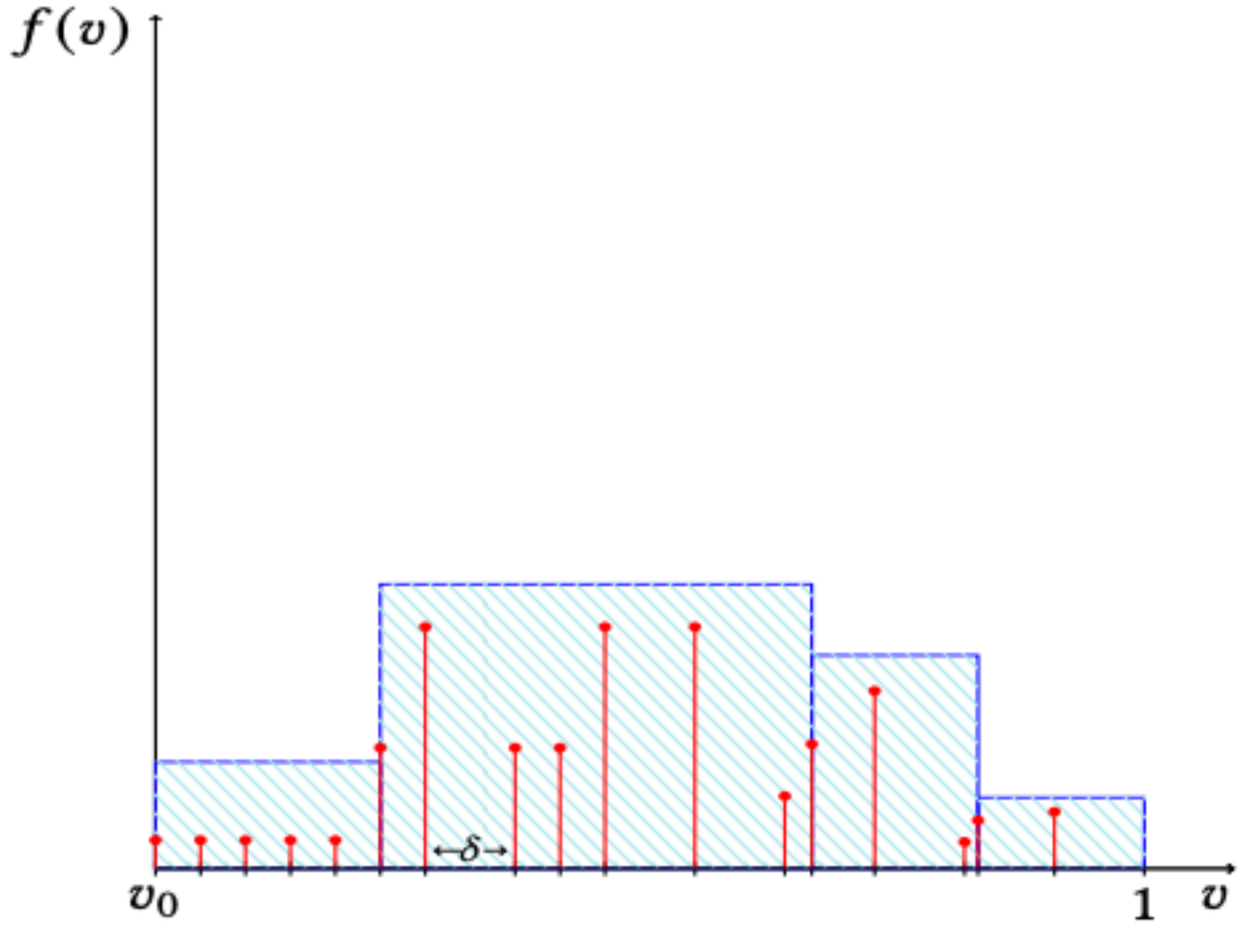

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PPAD-completeness

Theorem: [FGHK24] The problem of computing an ε-MBNE of a DFPA with subjective priors is PPAD-complete.

Proof Outline:

1. PPAD membership: We use our equivalence result to translate to the CFPA setting, which is in PPAD by [FGHLP23].

2. PPAD-hardness: Reduction from the PPAD-complete problem PURE-CIRCUIT [DFHM22].

Updated State

continuous priors

PBNE (trilateral tie-breaking):
PPAD-complete [CP23]

PBNE: PPAD- and FIXP-complete [FGHLP23]

iid priors | common priors | subjective priors

discrete priors

PBNE: NP-complete [FGHK24]

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- **Theorem:** [FGHK24] The problem of computing a symmetric ε-approximate MBNE of a DFPA with iid priors admits a PTAS.

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Proof Sketch

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3. Round the solution achieved in Step 2 so that it corresponds to a feasible set of strategies, provide a bound on the approximation factor of the MBNE.

Updated State/Future Work

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Thank you!

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References

1. [Ath01] Susan Athey. “Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information”. In: Econometrica 69.4 (July 2001), pp. 861–889.

2. [CP23] Xi Chen and Binghui Peng. “Complexity of Equilibria in First-Price Auctions under General Tie-Breaking Rules”. In: Proceedings of the 55th Annual ACM Symposium on Theory of Computing (STOC). 2023.

3. [DFHM22] Argyrios Deligkas, John Fearnley, Alexandros Hollender, and Themistoklis Melissourgos. “Pure-circuit: Strong inapproximability for PPAD”. In: Proceedings of the 63rd IEEE Symposium on Foundations of Computer Science (FOCS). 2022.

4. [EMS09] Guillaume Escamocher, Peter Bro Miltersen, and Rocio Santillan R. “Existence and Computation of Equilibria of First-Price Auctions with Integral Valuations and Bids”. In: Proceedings of The 8th International Conference on Autonomous Agents and Mutliagent Systems (AAMAS). 2009.

5. [FGHLP23] Aris Filos-Ratsikas, Yiannis Giannakopoulos, Alexandros Hollender, Philip Lazos, and Diogo Poças. “On the Complexity of Equilibrium Computation in First-Price Auctions”. In: SIAM Journal on Computing 52.1 (2023).

6. [FGHK24] Aris Filos-Ratsikas, Yiannis Giannakopoulos, Alexandros Hollender, Charalampos Kokkalis. “On the Computation of Equilibria in Discrete First-Price Auctions”. To appear in: Proceedings of the 25th ACM Conference on Economics and Computation (EC). 2024.

7. [GV88] D. Yu. Grigor’ev and N.N. Vorobjov. “Solving systems of polynomial inequalities in subexponential time”. In: Journal of Symbolic Computation 5.1-2 (1988).

Image References

1. https://www.ed.ac.uk/news/students/2014/summer-job
2. https://www.ed.ac.uk/history-classics-archaeology/news-events/events-archive/2019/considering-a-phd
3. https://www.independent.co.uk/travel/news-and-advice/edinburgh-festival-fringe-lonely-planet-uk-travelist-popular-experience-a9055171.html
4. https://economictimes.indiatimes.com/industry/miscellaneous/going-going-gone-covid-19-boosts-online-art-auctions/articleshow/78729989.cms?from=mdr
5. https://pubscale.com/blog/open-bidding-ebda